EASTERN UNIVERSITY, SRI LANKA
SECOND EXAMINATION IN SCIENCE 2000/2001
FIRST SEMESTER
Distribution Theory
ST 202.

Answer all questions

Time : 3 hours

Q1.

a).

i) State the properties of a Binomial experiment.

ii) An early warning detection system for aircraft consists of four identical radar units operating independently of one another. Suppose that each has a probability of 0.95 of detecting an intruding aircraft. When an intruding aircraft enters the scene, the random variable of interest is Y, the number of radar units that do not detect the plane. Is this a binomial experiment? Give the reasons.

b). Let X be a random variable with the probability density function

$$f_X(x;n,p)=\binom{n}{x}p^x q^{n-x} \qquad , \; x=0,1,2,\ldots\ldots\ldots,n.$$

i) Find the mean and variance of X.

ii) Show that,

$$f_x(x-1;n,p) < f_x(x;n,p) \text{ for } x<(n+1)p;$$

$$f_x(x-1;n,p) < f_x(x;n,p) \text{ for } x>(n+1)p; \text{and}$$

$$f_x(x-1;n,p) = f_x(x;n,p) \text{ if } x=(n+1)p \text{ and } (n+1)p \text{ is an integer.}$$

Q2.

a) A quality characteristic X of a manufactured item is a continuous random variable having probability density function

$$f(x)=\begin{cases}2\lambda^{-2}x & 0<x<\lambda\\ 0 & otherwise.\end{cases}$$

where λ is a positive constant whose value may be controlled by the manufacturer.

i) Find the mean and the variance of X in terms of λ.

ii) Every manufactured item is inspected before being dispatched for sale. Any item for which X is 8 or more is passed for selling and any item for which X is less than 8 is scrapped. The manufacturer makes a profit of Rs (27-λ) on every item passed for selling, and suffers a loss of Rs (λ+50) on every item that is scrapped. Find the value of λ which the manufacturer should aim for in order to maximize his expected profit per item, and calculate his maximum expected profit per item.

b) The random variable X has the probability density function

$$f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & othewise, \end{cases}$$

Find the moment generating function of X and hence find the mean and variance of X. Show also that the median of the distribution is ½ $\log_e 2$ and the inter-quartile range is ½ $\log_e 3$.

Q3.

a). The weights of randomly chosen packet of breakfast cereal A (including packing) may be taken to have a Normal distribution with mean 625g and standard deviation 15g. The weight of packaging may be taken to have an independent Normal distribution with mean 25g and Standard deviation 3g.

i) Find the probability that a randomly chosen packet of A has a total weight less than 630g.

ii) Find the probability that the total weight of the contents of four randomly chosen packets of A is less than 2450g.

iii) The weight of the contents of a randomly chosen packet of breakfast cereal B may be taken to have a Normal distribution with mean 465g and standard deviation 10g. Find the probability that the contents of four randomly chosen packets of B weigh more than the contents of three randomly chosen packets of A.

b). Suppose that $X_1, X_2, \ldots\ldots\ldots, X_{25}$ are 25 independent random variables each having distribution N(5,5). Explain how to create new random variables using one or more of the above random variables, so that the created variables have the following distributions.

1) N(0,1)
2) N(-5,7)
3) $\chi^2_{(1)}$
4) $\chi^2_{(5)}$
5) t_{20}
6) $F_{9,14}$

Q 4.

a). Let (X, Y) be two dimensional random variables;

1. Define the conditional expectation and conditional variance for X/Y.
2. Show that
   i. $E(Y) = E[E(Y/X)]$
   ii. $Var(Y) = E[Var[Y/X]] + Var[E[Y/X]]$.

b). Let the two dimensional random variable (X, Y) have the joint density

$$f_{xy}(x,y) = \begin{cases} \frac{1}{8}(6-x-y) & 0 < x < 2, \quad 2 < y < 4, \\ 0 & otherwise. \end{cases}$$

Find the followings:

1. $E[Y/X=x]$
2. $E[Y^2/X=x]$
3. $Var[Y/X=x]$
4. $E[XY/X=x]$.

Also show that $E[Y] = E[E[Y/X]]$.

Q 5.

Let X_1 and X_2 be two independent Standard Normal random variables. Also let

$Y_1 = X_1 + X_2$,
$Y2 = X_2 - X_1$,
$Y3 = \frac{1}{2} * (X_1 - X_2)^2$ and
$Y4 = X_1/X_2$.

1. Derive the joint probability density function of Y_1 and Y_2 using the moment generating function technique and find the marginal distribution of Y_1 and Y_2.

2. Derive the probability density function of Y_3 using the moment generating function technique.

3. Derive the joint probability density function of Y_1 and Y_4 using the transformation technique and find the marginal density of Y_4

Q6.

a). The probability density function of the gamma distribution is

$$f(x)=\begin{cases}\dfrac{\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)} & x>0,\\ 0 & otherwise,\end{cases}$$

where $\alpha, \lambda > 0$.

Show that the moment generating function of this distribution is $\left(\dfrac{\lambda}{\lambda-t}\right)^{\alpha}$ for $t<\lambda$ and hence find the mean and variance of the distribution.

b). A sample of n values is drawn from a population whose probability density function is

$$f(x)=\begin{cases}e^{-x} & x>0,\\ 0 & otherwise.\end{cases}$$

If $\overline{X}$ is the mean of the sample, show that $n\overline{X}$ has a Gamma distribution. What are the parameters of this distribution? Find the mean and variance of $\overline{X}$.

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