EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

EXTERNAL DEGREE EXAMINATION IN SCIENCE - 2004/2005

SECOND YEAR SECOND SEMESTER (Jan./Apr., 2010)

EXTMT 202 - ANALYSIS II (METRIC SPACE)

Answer all questions

Time: Two hours

1. (a) Define the term *metric space*.

   Show that the function $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ defined by

   $$d(x, y) = \min\{|x - y|, 1\} \text{ for all } x, y \in \mathbb{R} \text{ is a metric on } \mathbb{R}.$$

   (b) Let (X, d) be a metric space and (x_n) and (y_n) be sequences of points in X which converge to the points x and y in X , respectively. Prove that

   $$\lim_{n\to\infty} d(x_n, y_n) = d(x, y).$$

   (c) Show that, if a sequence is converges in a metric space (X, d) then it is a Cauchy sequence.

   Is the converse true? Justify your answer.

2. (a) Define the following terms in a metric space:

   i. *separated set*;

   ii. *disconnected set*.

   (b) Let (X, d) be a metric space. Prove that if $E \subseteq X$ is connected, then $\overline{E}$ is also connected.

   (c) Prove that, a metric space (X, d) is connected if and only if the only non empty subset of X which is both open and closed in X is itself.

(d) Let A be a connected subset of a metric space (X, d). Show that if B is a subset of X such that $A \subseteq B \subseteq \overline{A}$ then B is connected.

3. Define the term *compact set.*

(a) Show that the set $[a, b]$ is a compact subset of $\mathbb{R}$ with the usual metric.

(b) Let (X, d) be a compact metric space. Show that if F is a closed subset of X then F is compact.

(c) Prove that every compact subset of a metric space is bounded.

4. (a) What is meant by a function f from a metric space (X, d) to a metric space (Y, ρ) is continuous at $a \in X$?

Let (X, d) and (Y, d') be metric spaces and let $f : X \longrightarrow Y$ be a function and $x_0 \in X$. Prove that the following statements are equivalent.

i. For each open ball B with center at $f(x_0)$ there is an open ball B_0 with center at x_0 such that $B_0 \subseteq f^{-1}(B)$;

ii. For each open set U with $x_0 \in f^{-1}(U)$ there is an open ball B_0 with center at x_0 such that $B_0 \subseteq f^{-1}(U)$.

(b) Define the term *complete metric space.*

Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose that there is a bijection $f : X \to Y$ such that

$$\frac{1}{10} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq 10 d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$. Show that if X is complete, then Y must also be complete.