EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

EXTERNAL DEGREE EXAMINATION IN SCIENCE - 2008/2009

SECOND YEAR SECOND SEMESTER (Jan./ Apr., 2010)

EXT MT204 - ANALYSIS III

(RIEMANN INTEGRAL AND SEQUENCES AND SERIES OF FUNCTIONS)

Answer all questions Time : Two hours

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1. Let f be a bounded real valued function on $[a, b]$. Explain what is meant by the statement that "f is Riemann integrable over $[a, b]$".

   (a) With usual notations, prove that a bounded function f on $[a, b]$ is Riemann integrable if and only if for each $\epsilon > 0$ there is $\delta > 0$ depending on the choice of ϵ such that $\left| S(P, f, \zeta) - \int_a^b f(x)\, dx \right| < \epsilon$ for all partition P of $[a, b]$ with $\|P\| < \delta$ and for all selection of the intermediate points ζ.

   (b) Prove that if f is Riemann integrable over $[a, b]$ and there exist $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M,\ \forall x \in [a, b]$, then there exists $\mu \in [m, M]$ such that $\int_a^b f(x)\, dx = \mu(b - a)$.

2. (a) State what is meant by the statements "an improper integral of the first kind is convergent" and "an improper integral of the second kind is convergent"?

(b) Discuss the convergence of the improper integral $\int_a^b \frac{dx}{(x-a)^p}$, where p is a real number.

(c) Define the term "absolutely convergent" of an integral.

Prove that an absolutely convergent integral converges.

(d) Discuss the convergence of the followings:

i. $\int_1^\infty \frac{\cos x}{x^2}\, dx;$

ii. $\int_0^\infty \frac{1}{e^x+1}\, dx;$

iii. $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\, dx.$

3. Define the term "uniform convergence" of a sequence of functions.

(a) Prove that the sequence of real-valued functions $\{f_n\}_{n\in\mathbb{N}}$ defined on $E \subseteq \mathbb{R}$ converges uniformly on E if and only if for every $\epsilon > 0$ there exists an integer N such that $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in E$ and for all $m, n \geq N$.

(b) Let $\{f_n\}$ be a sequence of functions that are integrable on $[a, b]$ and suppose that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $[a, b]$. Prove that f is integrable and $\int_a^b f(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx.$

(c) Show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ where $f_n(x) = nxe^{-nx^2}$, $n \in \mathbb{N}$, converges but not uniformly on [0,1].

(a) Let $\{f_n\}_{n\in\mathbb{N}}$, $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of functions defined over a non-empty set $E \subseteq \mathbb{R}$. Suppose moreover that:

i. $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly in E;

ii. $\sum_{k=1}^{\infty} |g_{k+1}(x) - g_k(x)| \leq M$ for all $x \in E$, for some $M > 0$;

iii. $|g_1(x)| \leq M$ for all $x \in E$.

Prove that $\sum_{k=1}^{\infty} f_k(x) g_k(x)$ converges uniformly in E.

(b) Prove that $\sum_{k=1}^{\infty} \frac{\sin nx}{n}$ converges uniformly on $[\delta, \pi]$, where $\delta > 0$.