EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

THIRD EXAMINATION IN SCIENCE -2008/2009

SECOND SEMESTER (Feb./Mar., 2011)

EXTMT 303 - FUNCTIONAL ANALYSIS

EXTERNAL DEGREE

Answer all questions | Time: Two hours

1. Define the term *Banach space.*

   (a) If $\{x_1, x_2, \cdots, x_n\}$ is a set of linearly independent vectors in a normed linear space X, then there exists a number $k > 0$ such that $\| \sum_{i=1}^{n} \eta_i x_i \| \geq k \sum_{i=1}^{n} |\eta_i|$, for every choice of scalars $\eta_1, \eta_2, \cdots, \eta_n$.
   Use this result to prove the following:

      i. every finite dimensional subspace of X is complete,

      ii. any two norms on a finite dimensional normed linear space are equivalent.

   (b) Show that the sequence space

   $$l^1 = \left\{ x = (x_i) : x_i \in \mathbb{C}, \forall i \in \mathbb{N}, \sum_{i=1}^{\infty} |x_i| < \infty \right\}$$

   with the norm given by $\| x \| = \sum_{i=1}^{\infty} |x_i|$ is a Banach Space.

2. Define the term *bounded linear operator* from a normed linear space X into a normed linear space Y.

   Let T be a bounded linear operator from a normed linear space X into a normed linear space Y. Let

$\| T \|_1 = \inf\{M : \| T(x) \| \leq M \| x \|, \forall x \in X\}$;

$\| T \|_2 = \sup_{x \in X, \|x\| \leq 1} \| T(x) \|$;

$\| T \|_3 = \sup_{x \in X, \|x\| = 1} \| T(x) \|$ and

$\| T \| = \sup_{x \in X \setminus \{0\}} \frac{\| T(x) \|}{\| x \|}$.

Show that $\| T \| = \| T \|_1 = \| T \|_2 = \| T \|_3$.

3. State the *Hahn-Banach* theorem for normed linear spaces.

   (a) Let X be a normed linear space and let $x_0 \neq 0$ be any element of X. Prove that there exists a bounded linear functional g on X such that $\| g \| = 1$ and $g(x_0) = \| x_0 \|$.

   Deduce that if $f(x) = f(y)$ for every bounded linear functional f on X then $x = y$.

   (b) Let Y be a closed linear subspace of a normed linear space X and $x_0 \in X \setminus Y$, and let $\delta = \inf\{\| y - x_0 \| : y \in Y\}$. Show that there exists a bounded linear functional f defined on X such that $\| f \| = 1, f(Y) = \{0\}$ and $f(x_0) = \delta$.

4. (a) Define the term *linear functional.*

   Consider the normed linear space $C[a, b]$, set of all continuous functions defined on $[a, b]$, with norm given by $\| f \| = \sup_{a \leq t \leq b} | f(t) |$.

   Define $f : C[a, b] \to \mathbb{R}$ by $f(x) = \int_b^b x(t)dt, \ \forall\, x \in C[a, b]$.

   Show that:

   i. f is bounded and linear;

   ii. $\| f \| = (b - a)$.

   (b) Define the term *Schauder basis* in a normed linear space.

   Prove that the sequence $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis for the sequence space

   $l^p = \left\{ x = (x_i) : x_i \in \mathbb{C}, \forall i \in \mathbb{N}, \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$, where $1 \leq p < \infty$ and $e_i = (0, 0, \cdots, 1^{i^{th}}, 0, \cdots), i \in \mathbb{N}$.