EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

EXTERNAL DEGREE EXAMINATION IN SCIENCE - 2005/2006

SECOND YEAR, FIRST SEMESTER (Mar./May, 2010)

EXTMT 201 - VECTOR SPACES AND MATRICES

Answer all questions — Time: Three hours

1. Define the term *vector space*.

(a) Let V be a vector space over a field $\mathbb{F}$. Prove that a non-empty subset W of V is a subspace of V if and only if $\alpha x + \beta y \in W$, for any $x, y \in W$ and $\alpha, \beta \in \mathbb{F}$.

(b) Let $\mathbb{Q}(\sqrt{2}) = \{x + \sqrt{2}y : x, y \in \mathbb{Q}\}$, where $\mathbb{Q}$ denotes the set of all rational numbers. For any $X, Y \in \mathbb{Q}(\sqrt{2})$ the operations of addition $\oplus$ and a scalar multiplication $\odot$ are defined as follows:

$$X \oplus Y = (x_1 + y_1) + \sqrt{2}(x_2 + y_2)$$

and

$$\alpha \odot X = \alpha x_1 + \sqrt{2}\alpha x_2$$

where $X = x_1 + \sqrt{2}x_2,\ Y = y_1 + \sqrt{2}y_2$ and $\alpha \in \mathbb{Q}$.

Prove that $\left(\mathbb{Q}(\sqrt{2}), \oplus, \odot\right)$ is a vector space over $\mathbb{Q}$.

2. (a) Define the following:

i. a linearly independent set of vectors;

ii. a basis for a vector space;

iii. dimension of a vector space.

(b) Let V be an n-dimensional vector space. Show that

i. a linearly independent set of n vectors of V is a basis for V;

ii. any linearly independent set of vectors of V may be extended to a basis V;

iii. if L is a subspace of V, then there exists a subspace M of V such t $V = L \oplus M$, where $\oplus$ denote the direct sum.

(c) Let $\mathbb{P}_n = \left\{ \sum_{i=0}^{n} a_i x^i \ : \ a_i \in \mathbb{R}, n \in \mathbb{N} \right\}$ be the set of all polynomials of degree $\leq n$ with real coefficients.

i. If $S = \{2, x, x - x^2, x + x^2\}$ is a subset of $\mathbb{P}_2$, then find the dimension of (

ii. Show that $B = \{1, (x-1), (x-1)^2, (x-1)^3\}$ is a basis of $\mathbb{P}_3$. 5.

3. (a) Define the *range space* $R(T)$ and the *null space* $N(T)$ of a linear transformat T from a vector space V into another vector space W.

Find $R(T)$ and $N(T)$ of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3, x_2 + x_3, x_1 + x_2 - 2x_3).$$

Verify the equation $\dim V = \dim(R(T)) + \dim(N(T))$ for the above linear tra formation T.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(x, y, z) = (x + 2y, x + y + z, z)$ be a lin transformation and let $B_1 = \{(1, 1, 1), (1, 2, 3), (2, -1, 1)\}$ and $B_2 = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ be bases of $\mathbb{R}^3$.

i. Find the matrix representation of T with respect to the basis B_1;

ii. Using the transition matrix, find the matrix representation of T with resp to the basis B_2.

4. (a) Define the following terms:

(i) rank of a matrix;

(ii) row reduced echelon form of a matrix.

(b) Let A be an $m \times n$ matrix. Prove the following:

(i) row rank of A is equal to column rank of A;

(ii) if B is a matrix obtained by performing an elementary row operation on then A and B have the same rank.

(c) Find the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 3 & 0 & 2 \\ 2 & 1 & 3 & 3 & -1 & 3 \\ 2 & 1 & 1 & 1 & -2 & 4 \end{pmatrix}.$$

(d) Find the row reduced echelon form of the matrix

$$\begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix}.$$

5. Define the term *adjoint of* A as applied to an $n \times n$ matrix $A = (a_{ij})$.

(a) With the usual notations, prove that

$$A \cdot (adj A) = (adj A) \cdot A = det A \cdot I.$$

Hence prove the following:

i. $det(adj\ A) = (det\ A)^{n-1}$;

ii. $adj(adj\ A) = (det\ A)^{n-2} A$.

(b) Let P, Q and R be square matrices of the same order, where P and R are non-singular. Let O be the zero matrix of the same order. Prove that the inverse of the block matrix

$$\left(\begin{array}{c|c} P & O \\ \hline Q & R \end{array}\right)$$

is

$$\left(\begin{array}{c|c} P^{-1} & O \\ \hline -R^{-1}QP^{-1} & R^{-1} \end{array}\right).$$

(c) Find the determinant of

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$.

6. (a) State the necessary and sufficient condition for a system of linear equations to be consistent.

Reduce the augmented matrix of the following system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2. \end{aligned}$$

to its row reduced echelon form and hence determine the conditions on the non zero real numbers $a_{11}, a_{12}, a_{21}, a_{22}, b_1$ and b_2 such that the system has

(i) a unique solution;

(ii) no solution;

(iii) more than one solution.

(b) Find the condition on the real numbers b_1, b_2, b_3 and b_4 for the system of linear equations

$$\begin{aligned} x_1 - x_3 + 3x_4 + x_5 &= b_1 \\ 2x_1 + x_2 - 2x_4 - x_5 &= b_2 \\ x_1 + 2x_2 + 2x_3 + 4x_5 &= b_3 \\ x_2 + x_3 + 5x_4 + 6x_5 &= b_4 \end{aligned}$$

to be consistent.

Find the solution of the above system if $b_1 = -3, b_2 = 5, b_3 = 6$ and $b_4 = -2$.

(c) State and prove Crammer's rule for 3×3 matrix and use it to solve

$$\begin{aligned} 2x_1 - 5x_2 + 2x_3 &= 7 \\ x_1 + 2x_2 - 4x_3 &= 3 \\ 3x_1 - 4x_2 - 6x_3 &= 5. \end{aligned}$$