EASTERN UNIVERSITY, SRI LANKA

EXTERNAL DEGREE SECOND EXAMINATION IN SCIENCE(2002/2003)

SECOND SEMESTER (Oct./ Nov., 2007)

EXTMT 204 - RIEMANN INTEGRAL & SEQUENCES AND SERIES OF FUNCTIONS

Answer all questions **Time : Two hours**

1. Let f be a bounded real valued function on $[a, b]$. Explain what is meant by the statement that "f is Riemann integrable over $[a, b]$".

   (a) With usual notations, prove that a bounded function f on $[a, b]$ is Riemann integrable if and only if for each $\epsilon > 0$ there is $\delta > 0$ depending on the choice of ϵ such that $\left| S(P, f, \zeta) - \int_a^b f(x)\, dx \right| < \epsilon$ for all partition P of $[a, b]$ with $\|P\| < \delta$ and for all selection of the intermediate points ζ.

   (b) Let f be a Riemann integrable function on $[a, b]$. Prove that

   $$\int_a^b f(x)\, dx = \lim_{n\to\infty} \frac{(b-a)}{n} \sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right).$$

   Hence prove that

   $$\lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}\right] = \ln 2.$$

2. When is an integral $\int_a^b f(x)\,dx$ is said to be improper integral of the first kind, the second kind and the third kind?

What is meant by the statements "an improper integral of the first kind is convergent" and "an improper integral of the second kind is convergent"?

(a) Discuss the convergence of the improper integral $\int_a^b \frac{dx}{(x-a)^p}$.

(b) Discuss the convergence of the followings:

i. $\int_1^\infty \frac{\cos x}{x^2}\,dx$;

ii. $\int_0^\infty \frac{1}{e^x+1}\,dx$;

iii. $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$.

3. Define the term "uniform convergence of a sequence of functions".

(a) Prove that the sequence of real-valued functions $\{f_n\}_{n\in\mathbb{N}}$ defined on $E \subseteq \mathbb{R}$ converges uniformly on E if and only if for every $\epsilon > 0$ there exists an integer N such that $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in E$ and for all $m, n \geq N$.

(b) Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of differential real-valued functions on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some points $x_0 \in [a, b]$. Prove that if $\{f_n'\}_{n\in\mathbb{N}}$ converges uniformly on $[a, b]$, then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on $[a, b]$ to a differentiable function f, and $f'(x) = \lim_{n\to\infty} f_n'(x),\ \forall x \in [a, b]$.

4. (a) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued functions defined on $E \subseteq \mathbb{R}$. Suppose that for each $n \in \mathbb{N}$, there is a constant M_n such that

$$|f_n(x)| \leq M_n, \quad \text{for all } x \in E,$$

where $\sum_{k=1}^{\infty} M_n$ converges. Prove that $\sum_{k=1}^{\infty} f_n$ converges uniformly on E.

(b) Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of functions defined over a non empty set $E \subseteq \mathbb{R}$. Suppose also that

i. $|S_n| = \left|\sum_{k=1}^{n} f_k(x)\right| \leq M$ for all $x \in E$, and all $n \in \mathbb{N}$.

ii. $\sum_{k=1}^{\infty} |g_{k+1}(x) - g_k(x)|$ converges uniformly in E.

iii. $g_n \to 0$ unifomly in E as $n \to \infty$. Prove that $\sum_{k=1}^{\infty} f_k(x)\, g_k(x)$ converges uniformly in E.

(c) Show that $\sum_{k=1}^{\infty} \frac{(-1)^n}{n + ax^2}$ where $a > 0$ converges uniformly in $\mathbb{R}$.