EASTERN UNIVERSITY, SRI LANKA

EXTERNAL DEGREE EXAMINATION IN SCIENCE 2002/2003

SECOND YEAR SECOND SEMESTER(Oct./Dec.'2006)

EXMT 201 - VECTOR SPACES AND MATRICES

(Proper)

Answer All questions Time: Three hours

Q1. Define what is meant by a vector space.

(a) Let $M_{m\times n}$ be the set of all real $m \times n$ matrices. For any two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ in $M_{m\times n}$, and for any $\alpha \in \mathbb{R}$ define an addition $\oplus$ and scalar multiplication $\odot$ as follows:

$$[a_{ij}]_{m\times n} \oplus [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n},$$

$$\alpha \odot [a_{ij}]_{m\times n} = [\alpha a_{ij}]_{m\times n}$$

Prove that $(M_{m\times n}, \oplus, \odot)$ is a vector space over the field $\mathbb{R}$.

(b) Let W_1 and W_2 be two subspaces of a vector space V over a field $\mathbb{F}$ and let A_1 and A_2 be non-empty subsets of V. Show that

(i) $W_1 + W_2$ is the smallest subspace containing both W_1 and W_2,

(ii) if A_1 spans W_1 and A_2 spans W_2 then $A_1 \cup A_2$ spans $W_1 + W_2$.

(c) Let V be the vector space of all functions from real field $\mathbb{R}$ into $\mathbb{R}$. Which of the following subsets are subspaces of V? Justify your answer.

(i) $W_1 = \{f \in V : f(3) = 0\}$,

(ii) $W_2 = \{f \in V : f(7) = f(1)\}$,

(iii) $W_3 = \{f \in V : f(-x) = f(x), \forall x \in \mathbb{R}\}$.

Q2. (a) Define the following:

i. A linearly independent set of vectors,

ii. A basis for a vector space,

iii. Dimension of a vector space.

(b) Let V be an $n-$dimensional vector space. Show that

i. A linearly independent set of vectors of V with n elements is a basis for V,

ii. Any linearly independent set of vectors of V may be extended as a basis for V,

iii. If L is a subspace of V, then there exists a subspace M of V such that $V = L \oplus M$.

Q3. (a) Let T be a linear transformation from a vector space V into another vector space W. Define

(i) Range space $R(T)$,

(ii) Null space $N(T)$.

Find $R(T)$ and $N(T)$ of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(x, y, z) = (2x + y + 3z, 3x - y + z, -4x + 3y + z)$.

Verify the equation $\dim V = \dim(R(T)) + \dim(N(T))$ for this linear transformation.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y) = (x + 2y, 2x - y, -x)$ and let $B_1 = \{(0, 1), (1, 1)\}$ and $B_2 = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ be bases for $\mathbb{R}^3$. Find

(i) The matrix representation of T with respect to the basis B_1,

(ii) The matrix representation of T with respect to the basis B_2 by using the transition matrix,

(iii) The matrix representation of T with respect to the basis B_2 directly.

Q4. (a) Define the following terms

(i) Rank of a matrix,

(ii) Echelon form of a matrix,

(iii) Row reduced echelon form of a matrix.

(b) Let A be an $m \times n$ matrix. Prove that

(i) row rank of A is equal to column rank of A,

(ii) if B is an $m \times n$ matrix obtained by performing an elementary row operation on A, then $r(A) = r(B)$.

(c) Find the rank of the matrix

$$\begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{pmatrix}.$$

(d) Find the row reduced echelon form of the matrix

$$\begin{pmatrix} -1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix}.$$

Q5. (a) Define the following terms as applied to an $n \times n$ matrix $A = (a_{ij})$.

(i) Cofactor A_{ij} of an element a_{ij}

(ii) Adjoint of A.

Prove that

$$A \cdot (adj\, A) = (adj\, A) \cdot A = det A \cdot I,$$

where I is the $n \times n$ identity matrix.

(b) If $A = \begin{bmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & w & w^2 & w^3 \end{bmatrix}$, show that

$\det A = (x-y)(x-z)(x-w))(y-z)(y-w)(z-w)$.

(c) Find the inverse of the matrix

$$\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}.$$

Q6. (a) State the necessary and sufficient condition for a system of linear equations to be consistent.

Consider the following system of linear equations

$$\begin{aligned} ax + by &= e, \\ cx + dy &= f. \end{aligned}$$

Reduce the augmented matrix of the above system of linear equations to its row reduced echelon form and hence determine the conditions on a, b, c, d, e and f such that the system has

(i) a unique solution,

(ii) no solution,

(iii) more than one solution.

(b) State and prove Crammer's rule for 3×3 matrix and use it to solve

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ 3x_1 + 5x_2 - x_3 &= -5 \\ 2x_1 + x_2 + 2x_3 &= 5. \end{aligned}$$

(c) For what value of λ does the system

$$\begin{aligned} x + y + \quad t &= 4 \\ 2x \quad\quad - 4t &= 7 \\ x + y + z \quad\quad &= 5 \\ x - 3y - z - 10t &= \lambda \end{aligned}$$

has a solution. Find the general solution of the above system for this value of λ.