EASTERN UNIVERSITY, SRI LANKA

EXTERNAL DEGREE EXAMINATION IN SCIENCE

SECOND YEAR FIRST SEMESTER - 2002/2003

(Oct./Dec.' 2006)

EXTMT 203 - EIGENSPACES & QUADRATIC FORMS

Answer all questions Time : Two hours

1. Define the term "an eigenvalue of a linear transformation".

(a) Let A be a non singular matrix in $\mathbb{R}_{n\times n}$. Show that the characteristic polynomial of A^{-1} is

$$\chi_{A^{-1}}(t) = \frac{(-t)^n}{\det A}\, \chi_A\left(\frac{1}{t}\right), \qquad (t \neq 0).$$

Deduce that if $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the eigenvalues of A with algebraic multiplicities 1 then $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \cdots, \frac{1}{\alpha_n}$ are the eigenvalues of A^{-1} with algebraic multiplicities 1.

(b) Prove that an $n \times n$ matrix A is similar to diagonal matrix D whose diagonal elements are eigenvalues of A if and only if A has n linearly independent eigenvectors.

(c) Let

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Find a non-singular matrix P such that $P^{-1}AP$ is diagonal.

2. Define the terms "positive definite" and "orthogonal" as applied to a square matrix.

   (a) Prove that a matrix A is orthogonal if and only if columns of A form an orthonormal set.

   (b) Prove that a square matrix A is positive definite if and only if all the eigenvalues of A are positive.

   (c) Find an orthogonal matrix whose first column is $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

3. (a) Define the term "minimum polynomial" of a square matrix.

   (b) State and prove the Cayley-Hamilton theorem.

   By evaluating the characteristic polynomial of the matrix A given by

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 5 & -1 \end{pmatrix},$$

   show that $A^{-1} = -\frac{1}{12}(A^2 - 2A - 6I)$, where I is the identity matrix of order 3.

   (c) Find the minimum polynomial of the matrix A given by

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{pmatrix}.$$

4. (a) Prove that if λ_1 and λ_2 are two distinct roots of the equation $|A - \lambda B| = 0$, where A and B are real symmetric matrices and u_1 and u_2 are two vectors such that $(A - \lambda_i B)u_i = 0$ for $i = 1, 2$, then $u_1^T B u_2 = 0$.

   (b) Simultaneously diagonalize the following pair of quadratic forms

   $\phi_1 = x_1^2 - x_2^2 - 2x_3^2 - 2x_1x_2 + 4x_2x_3$,

   $\phi_2 = x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$.