EASTERN UNIVERSITY, SRI LANKA

EXTERNAL DEGREE EXAMINATION IN SCIENCE

FIRST YEAR FIRST SEMESTER - 2003/2004

(Oct./Dec.' 2006)

EXTMT 103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I

(Proper & Repeat)

Answer all questions | Time : Three hours

1. (a) For any three vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$, prove that the identity

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\ \underline{b} - (\underline{a} \cdot \underline{b})\ \underline{c}\,,$$

holds.

Hence prove that

$$(\underline{a} \wedge \underline{b}) \cdot [(\underline{b} \wedge \underline{c}) \wedge (\underline{c} \wedge \underline{a})] = [\underline{a} \cdot (\underline{b} \wedge \underline{c})]^2.$$

(b) The vectors $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$ are defined, in terms of vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ by

$$\underline{\alpha} = \frac{\underline{b} \wedge \underline{c}}{V}, \quad \underline{\beta} = \frac{\underline{c} \wedge \underline{a}}{V}, \quad \underline{\gamma} = \frac{\underline{a} \wedge \underline{b}}{V},$$

where $V = \underline{a} \cdot \underline{b} \wedge \underline{c} \neq 0$. Show that $\underline{\alpha} \cdot \underline{\beta} \wedge \underline{\gamma} = \dfrac{1}{V}$. Show also that any vector $\underline{r}$ can be expressed in the form

$$\underline{r} = (\underline{r} \cdot \underline{\alpha})\ \underline{a} + (\underline{r} \cdot \underline{\beta})\ \underline{b} + (\underline{r} \cdot \underline{\gamma})\ \underline{c}\,.$$

(c) If the vector $\underline{x}$ is given by the equation $\lambda \underline{x} + \underline{x} \wedge \underline{a} = \underline{b}$, where $\underline{a}$ and $\underline{b}$ are constant vectors and λ is a non-zero scalar, show that

$$\lambda^2(\underline{x} \wedge \underline{a}) + (\underline{a} \cdot \underline{b})\underline{a} - \lambda|\underline{a}|^2\underline{x} + \lambda(\underline{a} \wedge \underline{b}) = 0.$$

Hence obtain $\underline{x}$ in terms of $\underline{a}$, $\underline{b}$ and λ.

2. Define the following terms.

(a) A conservative vector field.

(b) The scalar potential.

If the force field $\underline{A} = \underline{\nabla}\phi$, where ϕ is a single valued and has continuous partial derivative, show that the work done by moving a particle from one point $P_1 \equiv (x_1, y_1, z_1)$ to another point $P_2 \equiv (x_2, y_2, z_2)$ in this field is independent of its path joining the two points.

Conversely, if $\int_C \underline{A} \cdot d\underline{r}$ is independent of the path C joining any two points, show that there exists a scalar function ϕ such that $\underline{A} = \underline{\nabla}\phi$.

Show that the field $\underline{A} = (\sin y + z)\underline{i} + (x \cos y - z)\underline{j} + (x - y)\underline{k}$ is conservative.

Obtain the field ϕ such that $\underline{A} = \underline{\nabla}\phi$ and hence evaluate $\int_A^B \underline{A} \cdot d\underline{r}$, where A is $(1, 0, 2)$ and B is $(2, \frac{\pi}{2}, 3)$.

3. State Green's theorem on the plane and Divergence theorem.

(a) Verify the Green's theorem in the plane for

$$\oint_C (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$$

where C is the closed curve of the region bounded by $y = x^2$ and $y = \sqrt{x}$.

(b) By using Divergence theorem or otherwise, evaluate $\int\int_S \underline{A} \cdot \underline{n}\, dS$, where $\underline{A} = 4xz\ \underline{i} - y^2\ \underline{j} + yz\ \underline{k}$ and S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$ and $z = 0$, $z = 1$.

4. A particle moves in a plane with velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. Prove that the components of acceleration of the particle along the tangent and perpendicular to it are $\frac{dv}{dt}$ and $v\frac{d\psi}{dt}$ respectively.

A body attached to a parachute is released from an aeroplane which is moving horizontally with velocity v_0. The parachute exerts a drag opposing motion which is k times the weight of the body, where k is a constant. Neglecting the air resistance to the motion of the body, prove that the velocity of the body when its path is inclined an angle ψ to the horizontal is

$$\frac{v_0 \sec\psi}{(\sec\psi + \tan\psi)^k}.$$

Prove that if $k = 1$ the body cannot have a vertical component of velocity greater than $\frac{v_0}{2}$.

5. A particle is projected horizontally along the inner surface of a smooth cone whose axis is vertical and vertex upwards. Find the pressure at any point in terms of the depth below the vertex. Show that the particle will leave the cone at the depth below the vertex given by

$$\left[\frac{V^2h^2}{g\tan^2\alpha}\right]^{\frac{1}{3}},$$

where h is the initial depth, V is the initial velocity and α is the semi angle of the cone.

6. With the usual notations, obtain the equation of the motion of the body whose mass increases with time in the following form

$$\underline{F} = m\frac{d\underline{v}}{dt} + (\underline{v} - \underline{u})\frac{dm}{dt}.$$

A rocket of total mass m contains fuel of mass ϵm $(0 < \epsilon < 1)$. This fuel burns at a constant rate k and the gas is ejected backward with the velocity u_0 relative to the rocket. Find the speed of the rocket when the fuel has been completely burnt. Find also the greatest height reached.