EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE (2002/2003)

EXTERNAL DEGREE

Sept./Auct. 2005

SECOND SEMESTER

MT 102 - ANALYSIS I

Answer all questions Time: Three hours

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1. (a) i. Define the terms "Supremum" and "Infimum" of a non-empty subset of $\mathbb{R}$.

   ii. State the completeness property of $\mathbb{R}$. [20]

   (b) Prove that an upper bound u of a non-empty bounded above subset S of $\mathbb{R}$ is the supremum of S if and only if for every $\epsilon > 0$, there exists $x \in S$ such that $x > u - \epsilon$.

   State the corresponding result for infimum. [30]

   (c) i. Let A and B be two non-empty bounded sets of real numbers. Let C be the set of all numbers $c = a + b$, where $a \in A,\ b \in B$.

   Prove that Sup C=Sup A + Sup B. [35]

   ii. Find the Supremum and Infimum of the set $\left\{\frac{2}{17}\left(1 - \frac{1}{11^n}\right) : n \in \mathbb{N}\right\}$, if they exist. [15]

2. (a) Define what is meant by each of the following terms applied to a sequence of real numbers.

i. bounded

ii. convergent

iii. monotone [30]

(b) Prove that, a monotone sequence (x_n) of real numbers is convergent if and only if it is bounded. [30]

(c) Let a sequence (x_n) be defined inductively by

$$x_1 = 4,\ x_{n+1} = \frac{1}{10}\left(x_n^2 + 21\right),\ n \in \mathbb{N}.$$

Show that

i. $3 < x_n < 7$ for all $n \in \mathbb{N}$.

ii. (x_n) is a decreasing sequence.

Deduce that (x_n) converges and find its limit. [40]

3. (a) i. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. What is meant by the function f has a limit $l \in \mathbb{R}$ at a point "a"$(\in \mathbb{R})$.

ii. Show that if $\lim_{x\to a} f(x) = l$, then $\lim_{x\to a} |f(x)| = |l|$.

Is the converse of this result true? Justify your answer. [35]

(b) i. Let $f : A(\subseteq \mathbb{R}) \to \mathbb{R}$, prove that $\lim_{x\to a} f(x) = l$ if and only if for every sequence (x_n) in A with $x_n \to a$ as $n \to \infty$ such that $x_n \neq a\ \forall\, n \in \mathbb{N}$, we have $f(x_n) \to l$ as $n \to \infty$. [35]

ii. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Show that the function g has a finite limit only at $x = 0$. [30]

4. (a) i. Write the (ϵ, δ) definition of the statement that $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point "a"$(\in \mathbb{R})$.

ii. Show that, if f is continuous at 'a' and $f(a) > 0$ then there exist some $\delta > 0$ such that $f(x) > \frac{f(a)}{2}$ for all x satisfying $|x - a| < \delta$.

[40]

(b) i. If $f : [a, b](\subseteq \mathbb{R}) \to \mathbb{R}$ is continuous on $[a, b]$ then prove that it is bounded on $[a, b]$.

Is the converse part true? Justify your answer. [40]

ii. Prove that function $f : \mathbb{R} \to \mathbb{R}$ defined by,

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

is not continuous at $x = 0$.

[20]

5. (a) State what is meant by the statement that a function $f : \mathbb{R} \to \mathbb{R}$ is

i. differential at $a(\in \mathbb{R})$,

ii. strictly decreasing at $a(\in \mathbb{R})$. [10]

(b) i. Prove that if a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ and $f'(a) < 0$, then f is strictly decreasing at a.

Is the converse true ? Justify your answer.

ii. Let a function $g : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$ such that $g'(a) = 0$ for some $a \in \mathbb{R}$. Suppose that $g''(a)$ exists. Prove that if $g''(a) > 0$, then g has a maximum at $x = a$. [90]

6. (a) Suppose that both real-valued functions f and g are continuous on $[a, b]$, differentiable on (a, b) and $g'(x) \neq 0 \ \forall \ x \in (a, b)$.

Prove that, for some $c \in (a, b)$,

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

If $f(d) = g(d) = 0$ for some $d \in (a, b)$, deduce that $\lim_{x \to d} \frac{f'(x)}{g'(x)} = \lim_{x \to d} \frac{f(x)}{g(x)}$.

[55]

(b) Evaluate the following limits

i. $\lim_{x \to 1} \left(\frac{1}{\log\, x} - \frac{1}{x - 1} \right)$

ii. $\lim_{x \to \infty} \left(x - \sqrt{1 + x^2} \right)$

iii. $\lim_{x \to \infty} x \log \left(1 + \frac{1}{x} \right)$.

[45]